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一类带有对数项的临界 Choquard 方程组的基态解*

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摘　要：考虑一类 Choquard 型耦合方程组，其中非线性项含有对数项和 Hardy-Littlewood-Sobolev 临界指数 . 当对数项的系数均为负值时，借助单个临界 Choquard 方程相应的局部极小点的存在性，建立了该系统对应能量泛函在 Nehari 流形中 Palais-Smale 序列的收敛性，进而利用 Ekeland 变分原理，获得了其具有极小能量的正解存在性 . 同时在对参数施加与线性问题相关的第一特征值的限制条件下，构造了上述系统具有负能量水平的非负解的存在性 . 本文的结果扩展了对数项系数为正值的情形，分析了系数的负性对能量泛函几何结构的影响，是对经典 Sobolev 临界系统在 Choquard 算子上的推广和延伸 .

关键词：Choquard 型系统；对数项；基态解；临界指数；耦合项

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Ground state solutions of a class of critical Choquard systems with logarithmic terms

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Abstract: A class of Choquard type coupled systems are considered, where Hardy-Littlewood-Sobolev critical exponents and logarithmic terms are contained in nonlinear terms. If the coefficients of logarithmic terms are both negative, Palais-Smale sequences of the energy functional corresponding to above problems in Nehari manifold are established by using of the existence on a local minima of single critical Choquard equation. Furthermore, by adopting Ekeland's variational principle, some restricted conditions under which the parameters are related to the first eigenvalue of linear operator with Dirichlet boundary conditions are given. The nonnegative solution with negative energy level of above systems is obtained. Our work generalizes the cases that the coefficients of logarithmic terms are positive, and analyzes the impact of negative coefficients on geometry structure of the energy functional. In fact, our results extend classical Sobolev critical systems to the corresponding Choquard problems.

Key words: Choquard systems; logarithmic terms; ground state solutions; critical exponents; coupled terms

本文研究下列带有对数项的 Choquard 型临界方程组

$$\begin{cases}-\Delta\phi=\mu_1\phi+\beta_1(I_\alpha*\phi^{2_\alpha^*})\phi^{2_\alpha^*-1}+\gamma(I_\alpha*\varphi^{2_\alpha^*})\phi^{2_\alpha^*-1}+\eta_1\phi\log\phi^2, & x\in\Omega,\\ -\Delta\varphi=\mu_2\varphi+\beta_2(I_\alpha*\varphi^{2_\alpha^*})\varphi^{2_\alpha^*-1}+\gamma(I_\alpha*\phi^{2_\alpha^*})\varphi^{2_\alpha^*-1}+\eta_2\varphi\log\varphi^2, & x\in\Omega,\\ \phi=\varphi=0, \quad x\in\partial\Omega,\end{cases} \tag{1}$$

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其中$\Omega \subset \mathbb{R}^N$为光滑有界域，$N \geqslant 3$，参数$\mu_1, \mu_2, \eta_1, \eta_2 \in \mathbb{R}$, $\beta_1, \beta_2 > 0$，$\gamma \neq 0$为耦合系数，$\alpha \in (0, N)$，I_α为Riesz势，且对于任意的$x \in \mathbb{R}^N \setminus \{0\}$，有$I_\alpha(x) = \frac{A_{N,\alpha}}{|x|^{N-\alpha}}$，其中$A_{N,\alpha} = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}}2^\alpha}$, $2_\alpha^* = \frac{N+\alpha}{N-2}$为Hardy-Littlewood-Sobolev型临界指数，卷积$f*g(x) = \int_\Omega f(x-y)g(y)\mathrm{d}y$.

近年来，带有卷积项的Choquard型问题受到了广泛关注(Ghimenti et al.，2016；Alves et al.，2021；Li，2023；Xia et al.，2021). 该算子出现在量子光学和单元素等离子体等诸多领域中. Gao et al.(2018)率先考虑了如下带有线性扰动的临界Choquard方程：

$$-\Delta u = \left(I_\alpha * |u|^{2_\alpha^*}\right)|u|^{2_\alpha^*-1} + \lambda u, \quad x \in \Omega, \tag{2}$$

并证明了当$\lambda < 0$时问题(2)的解均为平凡解，同时当$\lambda > 0$时，建立了问题(2)正解的存在性定理. 当问题(1)中不含有对数项时，即当$\eta_1 = \eta_2 = 0$时，You et al.(2019)研究了以下带有临界指数的耦合Choquard方程组：

$$\begin{cases} -\Delta\phi + \lambda_1\phi = \mu_1(I_\alpha*\phi^{2_\alpha^*})\phi^{2_\alpha^*-1} + \beta(I_\alpha*\varphi^{2_\alpha^*})\phi^{2_\alpha^*-1}, & x \in \Omega, \\ -\Delta\varphi + \lambda_2\varphi = \mu_2(I_\alpha*\varphi^{2_\alpha^*})\varphi^{2_\alpha^*-1} + \beta(I_\alpha*\phi^{2_\alpha^*})\varphi^{2_\alpha^*-1}, & x \in \Omega, \\ \phi,\ \varphi \geqslant 0, \quad x \in \Omega, \\ \phi = \varphi = 0, \quad x \in \partial\Omega, \end{cases}$$

其中$\mu_1, \mu_2 > 0$. 当$\beta > \frac{\alpha+2}{N-2}\max\{\mu_1, \mu_2\}$和$\beta < 0$，且满足条件$-\lambda_1(\Omega) < \lambda_1$，$\lambda_2 < 0$与$N \geqslant 5$时，借助Nehari流形的分解和截断函数的估计，给出了上述方程组存在基态解时的充分条件. 进一步，当$\beta \to -\infty$时，You 等也探讨了其正基态解的渐近性态. Zhang et al.(2023)考虑了带有预定质量的耦合临界Choquard方程组，利用Pohozaev流形的性质证明了当耦合系数充分大时正规解的存在性以及耦合系数充分小时正规解的不存在性. 此外，Giacomoni et al.(2018)和Zheng et al.(2020)均研究了带有临界指数的分数阶Choquard方程组正解的存在性与多重性，进而Geng et al.(2022)讨论了带有预定质量的耦合Choquard方程组正规解的存在性及渐近性.

另一方面，对于带有对数项的临界问题，已有丰富和深刻的结果(Li et al.，2023；Shuai et al.，2023；Zhang et al.，2023). Deng et al.(2023)研究了一类带有临界指数和对数项的Brezis-Nirenberg问题，当包含对数项的系数为负值时，证明了该问题具有山路型的基态解. 以此结果为基础，Hajaiej et al.(2024a)将含有对数项的单个方程推广到如下耦合方程组：

$$\begin{cases} -\Delta\phi = \lambda_1\phi + \mu_1|\phi|^2\phi + \beta|\varphi|^2\phi + \theta_1\phi\log\phi^2, & x \in \Omega, \\ -\Delta\varphi = \lambda_2\varphi + \mu_2|\varphi|^2\varphi + \beta|\phi|^2\varphi + \eta_2\varphi\log\varphi^2, & x \in \Omega, \\ \phi = \varphi = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

当β为正数且充分大，以及$|\beta|$充分小，在θ_1，$\theta_2 > 0$的情形下，建立了问题(3)极小能量解的存在性. 进而对于θ_1，$\theta_2 < 0$和$\theta_2 < 0 < \theta_1$的情形，在对参数μ_1和μ_2施加不同的限制条件下，分别给出了问题(3)具有局部极小点和非负解时的充分条件. 进一步，Hajaiej et al.(2024b)将边值问题(3)推广到如下高维情形下带有临界指数和对数项的椭圆系统：

$$\begin{cases} -\Delta\phi = \lambda_1\phi + \mu_1|\phi|^{2p-2}\phi + \beta|\phi|^{p-2}|\varphi|^p\phi + \theta_1\phi\log\phi^2, & x \in \Omega, \\ -\Delta\varphi = \lambda_2\varphi + \mu_2|\varphi|^{2p-2}\varphi + \beta|\phi|^p|\varphi|^{p-2}\varphi + \eta_2\varphi\log\varphi^2, & x \in \Omega, \\ \phi = \varphi = 0, & x \in \partial\Omega, \end{cases}$$

其中$\Omega \subset \mathbb{R}^N$为光滑有界域，$2p = \frac{2N}{N-2}$. 当$N \geqslant 5$，且$\beta, \lambda_i, \mu_i, \theta_i (i = 1, 2)$处于不同范围时，建立了上述问题正解的存在性与不存在性. 注意到当$N \geqslant 5$时，$2p \in (2, 4)$,而当$N = 4$时，$2p = 4$，将带来紧嵌入的新困难，需要发展新技术进行处理，本文正是受此技术的启发. 应当指出He et al.(2023)已将对数扰动项扩展

到如下带有临界指数的Choquard方程：

$$\begin{cases} -\Delta u = (I_\alpha * |u|^{2_\alpha^*})|u|^{2_\alpha^*-2}u + \lambda u \log u^2 + \beta u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{4}$$

当$\lambda > 0$，$\beta \in \mathrm{R}$且$N \geqslant 4$时，结合山路水平值的估计，证明了问题(4)具有一正解，且为古典解.

受上述文献(Deng et al.，2023；Hajaiej et al.，2024a；He et al.，2023)的激发，本文将带有Choquard算子的耦合项与对数项进行结合，研究临界方程组(1)，分别扩展了单个带有对数扰动的Choquard方程和不含对数扰动的Choquard方程组，且证明了问题(1)局部极小点的存在性，进而给出了问题(1)具有非负解时的充分条件，此时的能量泛函水平值为负值. Choquard型耦合项的出现，为验证收敛性带来障碍，借助Giacomoni et al.(2018)的积分不等式，克服了这一困难. 为证明Palais-Smale序列的弱极限即为非负解，利用山路几何结构获得了Palais-Smale序列的有界性.

定义乘积空间$\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega)$及与问题(1)对应的能量泛函$I$：$\mathcal{H} \to \mathbb{R}$如下：

$$\begin{aligned} I(\phi,\varphi) = &\frac{1}{2}\int_\Omega |\nabla\phi|^2 \mathrm{d}x - \frac{\mu_1}{2}\int_\Omega |\phi^+|^2 \mathrm{d}x - \frac{\beta_1}{2\cdot 2_\alpha^*}\int_\Omega \left(I_\alpha * |\phi^+|^{2_\alpha^*}\right)|\phi^+|^{2_\alpha^*}\mathrm{d}x - \frac{\eta_1}{2}\int_\Omega (\phi^+)^2(\log(\phi^+)^2 - 1)\mathrm{d}x \\ &+ \frac{1}{2}\int_\Omega |\nabla\varphi|^2 \mathrm{d}x - \frac{\mu_2}{2}\int_\Omega |\varphi^+|^2 \mathrm{d}x - \frac{\beta_2}{2\cdot 2_\alpha^*}\int_\Omega \left(I_\alpha * |\varphi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*}\mathrm{d}x - \frac{\eta_2}{2}\int_\Omega (\varphi^+)^2(\log(\varphi^+)^2 - 1)\mathrm{d}x \\ &- \frac{\gamma}{2_\alpha^*}\int_\Omega \left(I_\alpha * |\phi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*}\mathrm{d}x, \end{aligned}$$

其中$u^+ = \max\{u, 0\}$. 易见I在$\mathcal{H}$中定义合理. I的非负临界点与问题(1)的解相对应. 方便起见，记常数$\mathcal{C}_Y$为I在$\mathcal{H}$的子集Y中的下确界.

为给出我们的主要结果，定义以下集合：

$$\Upsilon_1 = \left\{ (\mu_1,\beta_1,\eta_1;\mu_2,\beta_2,\eta_2) \mid \mu_1,\mu_2 \in [0,\lambda_1(\Omega)),\ \beta_1,\beta_2 > 0,\ \eta_1,\eta_2 < 0, \right.$$

$$\left. \frac{\alpha+2}{2N+2\alpha}\left(\frac{(\min\{\lambda_1(\Omega)-\mu_1, \lambda_1(\Omega)-\mu_2\})^{2_\alpha^*}}{\lambda_1(\Omega)^{2_\alpha^*}\max\{\beta_1,\beta_2\}}\right)^{\frac{1}{2_\alpha^*-1}} S_{H,L}^{\frac{2_\alpha^*}{2_\alpha^*-1}} + \frac{(\eta_1+\eta_2)|\Omega|}{2} > 0 \right\},$$

$$\Upsilon_2 = \left\{ (\mu_1,\beta_1,\eta_1;\mu_2,\beta_2,\eta_2) \mid \mu_1 \in [0,\lambda_1(\Omega)),\ \mu_2 \in \mathbb{R},\ \beta_1,\beta_2 > 0,\ \eta_1,\eta_2 < 0, \right.$$

$$\left. \frac{\alpha+2}{2N+2\alpha}\left(\frac{(\lambda_1(\Omega)-\mu_1)^{2_\alpha^*}}{\lambda_1(\Omega)^{2_\alpha^*}\max\{\beta_1,\beta_2\}}\right)^{\frac{1}{2_\alpha^*-1}} S_{H,L}^{\frac{2_\alpha^*}{2_\alpha^*-1}} + \frac{\eta_1+\eta_2 e^{-\frac{\mu_2}{\eta_2}}}{2}|\Omega| > 0 \right\},$$

$$\Upsilon_3 = \left\{ (\mu_1,\beta_1,\eta_1;\mu_2,\beta_2,\eta_2) \mid \mu_1,\ \mu_2 \in \mathbb{R},\ \beta_1,\ \beta_2 > 0,\ \eta_1,\eta_2 < 0, \right.$$

$$\left. \frac{\alpha+2}{2N+2\alpha}\left(\frac{1}{\max\{\beta_1,\beta_2\}}\right)^{\frac{1}{2_\alpha^*-1}} S_{H,L}^{\frac{2_\alpha^*}{2_\alpha^*-1}} + \frac{\eta_1 e^{-\frac{\mu_1}{\eta_1}} + \eta_2 e^{-\frac{\mu_2}{\eta_2}}}{2}|\Omega| > 0 \right\},$$

其中$S_{H,L}$和$\lambda_1(\Omega)$分别由式(7)和式(8)给出.

定理1 设σ在引理1中给定，且$B_\sigma = \left\{ (\phi,\varphi) \in \mathcal{H} \,\middle|\, \sqrt{|\nabla\phi|_2^2 + |\nabla\varphi|_2^2} < \sigma \right\}$.

当$\gamma < 0$，若下列条件之一满足

(i) $(\mu_1,\beta_1,\eta_1;\mu_2,\beta_2,\eta_2) \in \Upsilon_1$,

(ii) $(\mu_1,\beta_1,\eta_1;\mu_2,\beta_2,\eta_2) \in \Upsilon_2$,

(iii) $(\mu_1,\beta_1,\eta_1;\mu_2,\beta_2,\eta_2) \in \Upsilon_3$.

当$\gamma > 0$，假设存在$\epsilon > 0$，使得下列条件之一成立

（iv）$\left(\mu_1, \beta_1 + \gamma\epsilon, \eta_1;\ \mu_2, \beta_2 + \dfrac{\gamma}{\epsilon}, \eta_2\right) \in \Upsilon_1$,

（v）$\left(\mu_1, \beta_1 + \gamma\epsilon, \eta_1;\ \mu_2, \beta_2 + \dfrac{\gamma}{\epsilon}, \eta_2\right) \in \Upsilon_2$,

（vi）$\left(\mu_1, \beta_1 + \gamma\epsilon, \eta_1;\ \mu_2, \beta_2 + \dfrac{\gamma}{\epsilon}, \eta_2\right) \in \Upsilon_3$,

则问题(1)具有一个正解$(\tilde{\phi}, \tilde{\varphi})$，满足$I(\tilde{\phi}, \tilde{\varphi}) = \mathcal{C}_{B_\sigma} < 0$，且为局部极小点.

定理2　记

$$E = \left\{(\phi, \varphi) \in \mathcal{H} \,\middle|\, I'(\phi, \varphi) = 0\right\}. \tag{5}$$

在定理1的条件下，进一步假设

$$\frac{N+\alpha}{\alpha+2}\min\{\eta_1, \eta_2\} \geqslant -\lambda_1(\Omega)\text{，或者 }\gamma > 0\text{，或者 }\gamma \in \left(-\sqrt{\beta_1\beta_2}, 0\right),$$

则问题(1)具有非负解$(\hat{\phi}, \hat{\varphi}) \neq (0,0)$，且满足$I(\hat{\phi}, \hat{\varphi}) = \mathcal{C}_E < 0$.

1　定理1的证明

令$2^* = \dfrac{2N}{N-2}$，$N \geqslant 3$. 我们的工作空间为$\mathcal{D}^{1,2}(\mathbb{R}^N)$，其定义如下

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{u \in L^{2^*}(\mathbb{R}^N) \,\middle|\, \nabla u \in L^2(\mathbb{R}^N)\right\}.$$

赋予其范数为$\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}$. 利用Sobolev嵌入不等式和Hardy-Littlewood-Sobolev不等式，分别定义嵌入常数S和$S_{H,L}$如下（Li，2023）

$$S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}}}, \tag{6}$$

$$S_{H,L} = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^N} \left(I_\alpha * |u|^{2_\alpha^*}\right)|u|^{2_\alpha^*} \mathrm{d}x\right)^{\frac{1}{2_\alpha^*}}}. \tag{7}$$

记$\lambda_1(\Omega)$为带有Dirichlet边界条件的算子$-\Delta$对应的第一特征值，即（Hajaiej et al.，2024a）

$$\lambda_1(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \mathrm{d}x}{\int_\Omega |u|^2 \mathrm{d}x}. \tag{8}$$

引理1　对于$\gamma < 0$，假设定理1中的条件(i)，(ii)及(iii)之一满足，对于$\gamma > 0$，假设存在$\epsilon > 0$，使得定理1中的条件(iv)，(v)及(vi)之一成立，则存在τ，$\sigma > 0$，使得对所有满足$\sqrt{|\nabla\phi|^2 + |\nabla\varphi|^2} = \sigma$的$(\phi, \varphi)$，有$I(\phi, \varphi) \geqslant \tau$.

证明　首先考虑$\gamma < 0$的情形.

$$\begin{aligned} I(\phi, \varphi) \geqslant\ & \frac{1}{2}\int_\Omega |\nabla\phi|^2 \mathrm{d}x - \frac{\mu_1}{2}\int_\Omega |\phi^+|^2 \mathrm{d}x - \frac{1}{2 \cdot 2_\alpha^*}\int_\Omega \beta_1\left(I_\alpha * |\phi^+|^{2_\alpha^*}\right)|\phi^+|^{2_\alpha^*} \mathrm{d}x \\ & - \frac{\eta_1}{2}\int_\Omega (\phi^+)^2(\log(\phi^+)^2 - 1)\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla\varphi|^2 \mathrm{d}x - \frac{\mu_2}{2}\int_\Omega |\varphi^+|^2 \mathrm{d}x \\ & - \frac{1}{2 \cdot 2_\alpha^*}\int_\Omega \beta_2\left(I_\alpha * |\varphi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*} \mathrm{d}x - \frac{\eta_2}{2}\int_\Omega (\varphi^+)^2(\log(\varphi^+)^2 - 1)\mathrm{d}x. \end{aligned}$$

进一步，分为以下3种情况进行讨论

（i）若$(\mu_1,\beta_1,\eta_1;\ \mu_2,\beta_2,\eta_2)\in\Upsilon_1$，则由式(7)~(8)可得

$$I(\phi,\varphi)\geqslant\frac{1}{2}\frac{\lambda_1(\Omega)-\mu_1}{\lambda_1(\Omega)}|\nabla\phi|_2^2-\frac{\beta_1}{2\cdot 2_\alpha^*}\frac{|\nabla\phi|_2^{2\cdot 2_\alpha^*}}{S_{H,L}^{2_\mu^*}}+\frac{\eta_1}{2}|\Omega|+\frac{1}{2}\frac{\lambda_1(\Omega)-\mu_2}{\lambda_1(\Omega)}|\nabla\varphi|_2^2-\frac{\beta_2}{2\cdot 2_\alpha^*}\frac{|\nabla\varphi|_2^{2\cdot 2_\alpha^*}}{S_{H,L}^{2_\alpha^*}}+\frac{\eta_2}{2}|\Omega|$$

$$\geqslant\frac{1}{2}\min\left\{\frac{\lambda_1(\Omega)-\mu_1}{\lambda_1(\Omega)},\ \frac{\lambda_1(\Omega)-\mu_2}{\lambda_1(\Omega)}\right\}\left(|\nabla\phi|_2^2+|\nabla\varphi|_2^2\right)-\frac{\max\{\beta_1,\ \beta_2\}}{2\cdot 2_\alpha^* S_{H,L}^{2_\alpha^*}}\left(|\nabla\phi|_2^2+|\nabla\varphi|_2^2\right)^{2_\alpha^*}+\frac{\eta_1+\eta_2}{2}|\Omega|.$$

取

$$\sigma=\left(\frac{\min\{\lambda_1(\Omega)-\mu_1,\lambda_1(\Omega)-\mu_2\}}{\lambda_1(\Omega)\max\{\beta_1,\ \beta_2\}}\right)^{\frac{1}{2\cdot 2_\alpha^*-2}}S_{H,L}^{\frac{2_\alpha^*}{2\cdot 2_\alpha^*-2}}$$

和

$$\tau=\frac{\alpha+2}{2N+2\alpha}\left(\frac{(\min\{\lambda_1(\Omega)-\mu_1,\lambda_1(\Omega)-\mu_2\})^{2_\alpha^*}}{\lambda_1(\Omega)^{2_\alpha^*}\max\{\beta_1,\ \beta_2\}}\right)^{\frac{1}{2_\alpha^*-1}}S_{H,L}^{\frac{2_\alpha^*}{2_\alpha^*-1}}+\frac{\eta_1+\eta_2}{2}|\Omega|,$$

可推出

$$I(\phi,\ \varphi)\geqslant\frac{\alpha+2}{2N+2\alpha}\left(\frac{(\min\{\lambda_1(\Omega)-\mu_1,\lambda_1(\Omega)-\mu_2\})^{2_\alpha^*}}{\lambda_1(\Omega)^{2_\alpha^*}\max\{\beta_1,\ \beta_2\}}\right)^{\frac{1}{2_\alpha^*-1}}S_{H,L}^{\frac{2_\alpha^*}{2_\alpha^*-1}}+\frac{\eta_1+\eta_2}{2}|\Omega|=\tau>0.$$

（ii）若$(\mu_1,\beta_1,\eta_1;\ \mu_2,\beta_2,\eta_2)\in\Upsilon_2$，则再根据式(7)~(8)可得

$$I(\phi,\ \varphi)\geqslant\frac{1}{2}\frac{\lambda_1(\Omega)-\mu_1}{\lambda_1(\Omega)}|\nabla\phi|_2^2-\frac{\beta_1}{2\cdot 2_\alpha^*}\frac{|\nabla\phi|_2^{2\cdot 2_\alpha^*}}{S_{H,L}^{2_\alpha^*}}+\frac{\eta_1}{2}|\Omega|+\frac{1}{2}|\nabla\varphi|_2^2-\frac{\beta_2}{2\cdot 2_\alpha^*}\frac{|\nabla\varphi|_2^{2\cdot 2_\alpha^*}}{S_{H,L}^{2_\alpha^*}}+\frac{\eta_2 e^{-\frac{\mu_2}{\eta_2}}}{2}|\Omega|$$

$$\geqslant\frac{1}{2}\frac{\lambda_1(\Omega)-\mu_1}{\lambda_1(\Omega)}\left(|\nabla\phi|_2^2+|\nabla\varphi|_2^2\right)-\frac{\max\{\beta_1,\ \beta_2\}}{2\cdot 2_\alpha^* S_{H,L}^{2_\alpha^*}}\left(|\nabla\phi|_2^2+|\nabla\varphi|_2^2\right)^{2_\alpha^*}+\frac{\eta_1+\eta_2 e^{-\frac{\mu_2}{\eta_2}}}{2}|\Omega|.$$

取

$$\sigma=\left(\frac{\lambda_1(\Omega)-\mu_1}{\lambda_1(\Omega)\max\{\beta_1,\beta_2\}}\right)^{\frac{1}{2\cdot 2_\alpha^*-2}}S_{H,L}^{\frac{2_\alpha^*}{2\cdot 2_\alpha^*-2}}$$

和

$$\tau=\frac{\alpha+2}{2N+2\alpha}\left(\frac{(\lambda_1(\Omega)-\mu_1)^{2_\alpha^*}}{\lambda_1(\Omega)^{2_\alpha^*}\max\{\beta_1,\ \beta_2\}}\right)^{\frac{1}{2_\alpha^*-1}}S_{H,L}^{\frac{2_\alpha^*}{2_\alpha^*-1}}+\frac{\eta_1+\eta_2 e^{-\frac{\lambda_2}{\eta_2}}}{2}|\Omega|,$$

我们得到

$$I(\phi,\ \varphi)\geqslant\frac{\alpha+2}{2N+2\alpha}\left(\frac{(\lambda_1(\Omega)-\mu_1)^{2_\alpha^*}}{\lambda_1(\Omega)^{2_\alpha^*}\max\{\beta_1,\ \beta_2\}}\right)^{\frac{1}{2_\alpha^*-1}}S_{H,L}^{\frac{2_\alpha^*}{2_\alpha^*-1}}+\frac{\eta_1+\eta_2 e^{-\frac{\lambda_2}{\eta_2}}}{2}|\Omega|=\tau>0.$$

（iii）若$(\mu_1,\beta_1,\eta_1;\ \mu_2,\beta_2,\eta_2)\in\Upsilon_3$，运用式(7)~(8)，推出

$$I(\phi,\ \varphi)\geqslant\frac{1}{2}|\nabla\phi|_2^2-\frac{\beta_1}{2\cdot 2_\alpha^*}\frac{|\nabla\phi|_2^{2\cdot 2_\alpha^*}}{S_{H,L}^{2_\mu^*}}+\frac{\eta_1 e^{-\frac{\mu_1}{\eta_1}}}{2}|\Omega|+\frac{1}{2}|\nabla\varphi|_2^2-\frac{\beta_2}{2\cdot 2_\alpha^*}\frac{|\nabla\varphi|_2^{2\cdot 2_\alpha^*}}{S_{H,L}^{2_\alpha^*}}+\frac{\eta_2 e^{-\frac{\mu_2}{\eta_2}}}{2}|\Omega|$$

$$\geqslant\frac{1}{2}\left(|\nabla\phi|_2^2+|\nabla\varphi|_2^2\right)-\frac{\max\{\beta_1,\ \beta_2\}}{2\cdot 2_\alpha^* S_{H,L}^{2_\alpha^*}}\left(|\nabla\phi|_2^2+|\nabla\varphi|_2^2\right)^{2_\alpha^*}+\frac{\eta_1 e^{-\frac{\mu_1}{\eta_1}}+\eta_2 e^{-\frac{\mu_2}{\eta_2}}}{2}|\Omega|.$$

选取

$$\sigma=\left(\frac{S_{H,L}^{2_\alpha^*}}{\max\{\beta_1,\ \beta_2\}}\right)^{\frac{1}{2\cdot 2_\alpha^*-2}}$$

和

$$\tau=\frac{\alpha+2}{2N+2\alpha}\left(\frac{S_{H,L}^{2_\alpha^*}}{\max\{\beta_1,\beta_2\}}\right)^{\frac{1}{2_\alpha^*-1}}+\frac{\eta_1 e^{-\frac{\mu_1}{\eta_1}}+\eta_2 e^{-\frac{\mu_2}{\eta_2}}}{2}|\Omega|,$$

推出

$$I(\phi,\varphi)\geqslant\frac{\alpha+2}{2N+2\alpha}\left(\frac{S_{H,L}^{2_\alpha^*}}{\max\{\beta_1,\beta_2\}}\right)^{\frac{1}{2_\alpha^*-1}}+\frac{\eta_1 e^{-\frac{\mu_1}{\eta_1}}+\eta_2 e^{-\frac{\mu_2}{\eta_2}}}{2}|\Omega|=\tau>0.$$

接下来，考虑$\gamma>0$的情形.

由于

$$\gamma\int_\Omega\left(I_\alpha*|\phi_n^+|^{2_\alpha^*}\right)|\varphi_n^+|^{2_\alpha^*}dx\leqslant\frac{\gamma}{2}\left(\int_\Omega\epsilon\left(I_\alpha*|\phi_n^+|^{2_\alpha^*}\right)|\phi_n^+|^{2_\alpha^*}dx+\int_\Omega\frac{1}{\epsilon}\left(I_\alpha*|\varphi_n^+|^{2_\alpha^*}\right)|\varphi_n^+|^{2_\alpha^*}dx\right),$$

有

$$\begin{aligned}I(\phi,\varphi)\geqslant&\frac{1}{2}\int_\Omega|\nabla\phi|^2dx-\frac{\mu_1}{2}\int_\Omega|\phi^+|^2dx-\frac{\beta_1+\gamma\epsilon}{2\cdot 2_\alpha^*}\int_\Omega\left(I_\alpha*|\phi^+|^{2_\alpha^*}\right)|\phi^+|^{2_\alpha^*}dx\\&-\frac{\eta_1}{2}\int_\Omega(\phi^+)^2(\log(\phi^+)^2-1)dx+\frac{1}{2}\int_\Omega|\nabla\varphi|^2dx-\frac{\mu_2}{2}\int_\Omega|\varphi^+|^2dx\\&-\frac{\beta_2+\frac{\gamma}{\epsilon}}{2\cdot 2_\alpha^*}\int_\Omega\left(I_\alpha*|\varphi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*}dx-\frac{\eta_2}{2}\int_\Omega(\varphi^+)^2(\log(\varphi^+)^2-1)dx.\end{aligned}$$

类似于$\gamma<0$的情形，完成证明.

引理2　若引理1的条件满足，且$\mathcal{C}_{B_\sigma}$由定理1给出，则$-\infty<\mathcal{C}_{B_\sigma}<0$.

证明　当$|\nabla\phi|_2^2+|\nabla\varphi|_2^2<\sigma^2$时，易见$I(\phi,\varphi)>-\infty$，进而得到$\mathcal{C}_{B_\sigma}>-\infty$. 我们再来证明$\mathcal{C}_{B_\sigma}<0$. 注意到$\eta_1<0$. 固定$(\phi,0)$满足$|\nabla\phi|_2<\sigma$. 对于$t<1$，可得

$$\begin{aligned}I(t\phi,0)=t^2\Bigg(&\frac{1}{2}\int_\Omega|\nabla\phi|^2dx-\frac{\mu_1}{2}\int_\Omega|\phi^+|^2dx-\frac{\beta_1}{2\cdot 2_\alpha^*}t^{2\cdot 2_\alpha^*-2}\int_\Omega\left(|I_\alpha*|\phi^+|^{2_\alpha^*}\right)|\phi^+|^{2_\alpha^*}dx\\&-\frac{\eta_1}{2}\int_\Omega(\phi^+)^2(\log(\phi^+)^2-1)dx-\eta_1\log t\int_\Omega(\phi^+)^2dx\Bigg).\end{aligned}$$

选取ϕ满足$\int_\Omega(\phi^+)^2dx>0$. 则对充分小的$t$，有$I(t\phi,0)<0$，得证.

对于任意$c\in\mathbb{R}$，当$n\to\infty$时，若有$I(\phi_n,\varphi_n)\to c$且$I'(\phi_n,\varphi_n)\to 0$，则称$\{(\phi_n,\varphi_n)\}$为$I$在$\mathcal{D}^{1,2}(\mathbb{R}^N)$上的$(PS)_c$序列.

引理3　若$\{(\phi_n,\varphi_n)\}$为$(PS)_c$序列，且$\eta_i<0, i=1,2$. 则$\{(\phi_n,\varphi_n)\}$在$\mathcal{H}$中有界.

证明　由于$I'(\phi_n,\varphi_n)(\phi_n,\varphi_n)=o_n\left(|\nabla\phi_n|_2+|\nabla\varphi_n|_2\right)$，故对充分大的$n$，有

$$\begin{aligned}c+|\nabla\phi_n|_2+|\nabla\varphi_n|_2+1\geqslant&\ I(\phi_n,\varphi_n)-\frac{1}{2\cdot 2_\alpha^*}I'(\phi_n,\varphi_n)(\phi_n,\varphi_n)\\=&\frac{\alpha+2}{2N+2\alpha}|\nabla\phi_n|_2^2-\frac{\alpha+2}{2N+2\alpha}\eta_1\int_\Omega(\phi_n^+)^2\log\left(e^{\frac{\mu_1}{\eta_1}-\frac{N+\alpha}{\alpha+2}}(\phi_n^+)^2\right)dx\\&+\frac{\alpha+2}{2N+2\alpha}|\nabla\varphi_n|_2^2-\frac{\alpha+2}{2N+2\alpha}\eta_2\int_\Omega(\varphi_n^+)^2\log\left(e^{\frac{\mu_2}{\eta_2}-\frac{N+\alpha}{\alpha+2}}(\varphi_n^+)^2\right)dx.\end{aligned}$$

既然$\eta_1<0$，运用不等式$t\log t\geqslant -e^{-1}$，可导出

$$\begin{aligned}\frac{\alpha+2}{2N+2\alpha}\eta_1\int_\Omega(\phi_n^+)^2\log\left(e^{\frac{\mu_1}{\eta_1}-\frac{N+\alpha}{\alpha+2}}(\phi_n^+)^2\right)dx\leqslant&\frac{\alpha+2}{2N+2\alpha}\eta_1\int_{e^{\frac{\mu_1}{\eta_1}-\frac{N+\alpha}{\alpha+2}}(\phi_n^+)^2\leqslant 1}(\phi_n^+)^2\log\left(e^{\frac{\mu_1}{\eta_1}-\frac{N+\alpha}{\alpha+2}}(\phi_n^+)^2\right)dx\\\leqslant&-\frac{\alpha+2}{2N+2\alpha}\eta_1 e^{\frac{N+\alpha}{\alpha+2}-\frac{\mu_1}{\eta_1}}\int_{e^{\frac{\mu_1}{\eta_1}-\frac{N+\alpha}{\alpha+2}}(\phi_n^+)^2\leqslant 1}e^{-1}dx\leqslant-\frac{\alpha+2}{2N+2\alpha}\eta_1 e^{\frac{N-2}{\alpha+2}-\frac{\mu_1}{\eta_1}}|\Omega|.\end{aligned}$$

类似地，由于$\eta_2<0$，可得

$$\frac{\alpha+2}{2N+2\alpha}\eta_2\int_\Omega(\varphi_n^+)^2\log\left(e^{\frac{\mu_2}{\eta_2}-\frac{N+\alpha}{\alpha+2}}(\varphi_n^+)^2\right)dx\leqslant-\frac{\alpha+2}{2N+2\alpha}\eta_2 e^{\frac{N-2}{\alpha+2}-\frac{\mu_2}{\eta_2}}|\Omega|.$$

对充分大的n，有

$$c+|\nabla\phi_n|_2+|\nabla\varphi_n|_2+1\geqslant\frac{\alpha+2}{2N+2\alpha}|\nabla\phi_n|_2^2+\frac{\alpha+2}{2N+2\alpha}|\nabla\varphi_n|_2^2+\frac{\alpha+2}{2N+2\alpha}\eta_1 e^{\frac{N-2}{\alpha+2}-\frac{\mu_1}{\eta_1}}|\Omega|+\frac{\alpha+2}{2N+2\alpha}\eta_2 e^{\frac{N-2}{\alpha+2}-\frac{\mu_2}{\eta_2}}|\Omega|.$$

由此导出$\{(\phi_n,\varphi_n)\}$在$\mathcal{H}$中是有界的.

定理1的证明 借助引理1，对于$\mathcal{C}_{B_\sigma}$，我们选取极小化序列$\{(\phi_n,\varphi_n)\}\subset B_{\sigma-\delta}$，其中$\delta>0$且足够小. 利用Ekeland变分原理（Willem，1996），可假设$I'(\phi_n,\varphi_n)\to 0$. 由引理3，可知$\{(\phi_n,\varphi_n)\}$在$\mathcal{H}$中有界. 因此，我们假设

$$(\phi_n,\varphi_n)\rightharpoonup(\phi,\varphi)\quad 在\ \mathcal{H}中.$$

不失一般性，至多选取子列，根据式(6)，可假设

$$\begin{cases}\phi_n\rightharpoonup\phi,\ \ \varphi_n\rightharpoonup\varphi & 在\ L^{2^*}(\Omega)中,\\ \phi\to\phi,\ \ \varphi_n\to\varphi & 在\ L^2(\Omega)中,\\ \phi_n(x)\xrightarrow{a.e.}\phi(x),\ \ \varphi_n(x)\xrightarrow{a.e.}\varphi(x) & 在\ \Omega中.\end{cases}$$

由$|\phi_n|^{2_\alpha^*}\rightharpoonup|\phi|^{2_\alpha^*}$, $|\varphi_n|^{2_\alpha^*}\rightharpoonup|\varphi|^{2_\alpha^*}$在$L^{\frac{2N}{N+\alpha}}(\Omega)$中，可得$|\phi_n|^{2_\alpha^*-2}\phi_n\rightharpoonup|\phi|^{2_\alpha^*-2}\phi$，$|\varphi_n|^{2_\alpha^*-2}\varphi_n\rightharpoonup|\varphi|^{2_\alpha^*-2}\varphi$在$L^{\frac{2N}{\alpha+2}}(\Omega)$中. 进而结合Hardy-Littlewood-Sobolev不等式，及Riesz势表示由$L^{\frac{2N}{N+\alpha}}(\Omega)$映入$L^{\frac{2N}{N-\alpha}}(\Omega)$的连续线性算子（You et al.,2019），推出

$$I_\alpha*|\phi_n|^{2_\alpha^*}\rightharpoonup I_\alpha*|\phi|^{2_\alpha^*},\ \ I_\alpha*|\varphi_n|^{2_\alpha^*}\rightharpoonup I_\alpha*|\varphi|^{2_\alpha^*}\ 在\ L^{\frac{2N}{N-\alpha}}(\Omega)中.$$

进而可得

$$\left(I_\alpha*|\phi_n|^{2_\alpha^*}\right)|\phi_n|^{2_\alpha^*-2}\phi_n\rightharpoonup\left(I_\alpha*|\phi|^{2_\alpha^*}\right)|\phi|^{2_\alpha^*-2}\phi,\ \left(I_\alpha*|\varphi_n|^{2_\alpha^*}\right)|\varphi_n|^{2_\alpha^*-2}\varphi_n\rightharpoonup\left(I_\alpha*|\varphi|^{2_\alpha^*}\right)|\varphi|^{2_\alpha^*-2}\varphi\ \ 在\ L^{\frac{2N}{N+2}}(\Omega)中.$$

由范数的弱下半连续性，可得$(\phi,\varphi)\in B_\sigma$. 由不等式$|t^2\log t^2|\leqslant Ct^{2-p}+Ct^{2+p}$, $p\in(0,1)$（He et al.,2023），并运用Lebesgue控制收敛定理，对于任意的$\psi\in C_0^\infty(\Omega)$，可得

$$\lim_{n\to\infty}\int_\Omega\phi_n^+\psi^+\log(\phi_n^+)^2dx=\int_\Omega\phi^+\psi^+\log(\phi^+)^2dx.$$

故$I'(\phi,\varphi)=0$,且$\phi\geqslant 0$，$\varphi\geqslant 0$. 令$u_n=\phi_n-\phi$并且$v_n=\varphi_n-\varphi$. 根据Choquard型Brezis-Lieb引理，可得

$$\int_\Omega\left(I_\alpha*|\phi_n^+|^{2_\alpha^*}\right)|\phi_n^+|^{2_\alpha^*}dx=\int_\Omega\left(I_\alpha*|\phi^+|^{2_\alpha^*}\right)|\phi^+|^{2_\alpha^*}dx+\int_\Omega\left(I_\alpha*|u_n^+|^{2_\alpha^*}\right)|u_n^+|^{2_\alpha^*}dx+o_n(1),$$

$$\int_\Omega\left(I_\alpha*|\varphi_n^+|^{2_\alpha^*}\right)|\varphi_n^+|^{2_\alpha^*}dx=\int_\Omega\left(I_\alpha*|\varphi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*}dx+\int_\Omega\left(I_\alpha*|v_n^+|^{2_\alpha^*}\right)|v_n^+|^{2_\alpha^*}dx+o_n(1),$$

$$\int_\Omega\left(I_\alpha*|\phi_n^+|^{2_\alpha^*}\right)|\varphi_n^+|^{2_\alpha^*}dx=\int_\Omega\left(I_\alpha*|\phi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*}dx+\int_\Omega\left(I_\alpha*|u_n^+|^{2_\alpha^*}\right)|v_n^+|^{2_\alpha^*}dx+o_n(1),$$

其中$o_n(1)$表示无穷小量. 进而推出

$$|\nabla u_n|_2^2=\beta_1\int_\Omega\left(I_\alpha*|u_n^+|^{2_\alpha^*}\right)|u_n^+|^{2_\alpha^*}dx+o_n(1),\qquad|\nabla v_n|_2^2=\beta_2\int_\Omega(I_\alpha*|v_n^+|^{2_\alpha^*})^{2_\alpha^*}dx+o_n(1),$$

$$I(\phi_n,\varphi_n)=I(\phi,\varphi)+\frac{\alpha+2}{2N+2\alpha}\int_\Omega|\nabla u_n|^2dx+\frac{\alpha+2}{2N+2\alpha}\int_\Omega|\nabla v_n|^2dx+o_n(1).\tag{9}$$

通过选取子列，我们可假设

$$\int_\Omega|\nabla u_n|^2dx=l_1+o_n(1),\qquad\int_\Omega|\nabla v_n|^2dx=l_2+o_n(1).$$

在式(9)中令$n\to\infty$，有

$$\mathcal{C}_\sigma\leqslant I(\phi,\varphi)\leqslant I(\phi,\varphi)+\frac{\alpha+2}{2N+2\alpha}l_1+\frac{\alpha+2}{2N+2\alpha}l_2=\lim_{n\to\infty}I(\phi_n,\varphi_n)=\mathcal{C}_\sigma,$$

从而导出$l_1=l_2=0$.因此，再次选取子列，可得

$$(\phi_n,\varphi_n)\to(\phi,\varphi)\quad 在\ \mathcal{H}中.$$

既然$I(\phi,\varphi)=\mathcal{C}_{B_\sigma}<0$，我们有$(\phi,\varphi)\neq(0,0)$.

现证(ϕ,φ)不是半平凡的. 采用反证法进行证明，若不然，我们假设$\phi\equiv 0$，$\varphi\neq 0$. 对于充分小的$t>0$，考察$I(t\psi,\varphi)$，对于$(t\psi,\varphi)\in B_\sigma$，有

$$I(t\psi,\ \varphi)=I(0,\ \varphi)+t^2\left(\frac{1}{2}\int_\Omega|\nabla\psi|^2\mathrm{d}x-\frac{\mu_1}{2}\int_\Omega|\psi^+|^2\mathrm{d}x-\frac{\beta_1}{2\cdot 2_\alpha^*}t^{2\cdot 2_\alpha^*-2}\int_\Omega\left(I_\alpha*|\psi^+|^{2_\alpha^*}\right)|\psi^+|^{2_\alpha^*}\mathrm{d}x\right.$$
$$\left.-\frac{\eta_1}{2}\int_\Omega(\psi^+)^2(\log(\psi^+)^2-1)\mathrm{d}x-\eta_1\log t\int_\Omega(\psi^+)^2\mathrm{d}x-\frac{\gamma}{2_\alpha^*}t^{2_\alpha^*-2}\int_\Omega\left(I_\alpha*|\psi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*}\mathrm{d}x\right).$$

选择ψ使得$\int_\Omega(\psi^+)^2\mathrm{d}x>0$. 则对于充分小的$t$，有$I(t\psi,\varphi)<I(0,\varphi)=\mathcal{C}_{B_\sigma}$，与$\mathcal{C}_{B_\sigma}$的定义矛盾. 故$\phi\neq 0$，$\varphi\neq 0$. 借助Choquard算子的强极值原理(Cingolani et al.,2022)，可得ϕ，$\varphi>0$. 得证.

2　定理2的证明

引理4　在引理1的条件下，进一步假设$\frac{N+\alpha}{\alpha+2}\min\{\eta_1,\eta_2\}\geqslant-\lambda_1(\Omega)$，或者$\gamma>0$，或者$\gamma\in(-\sqrt{\beta_1\beta_2},0)$. 则$-\infty<\mathcal{C}_E<0$.

证明　由式(5)，注意到$(\phi,\varphi)\in E$，其中(ϕ,φ)是由定理1给定的解，导出$\mathcal{C}_E\leqslant\mathcal{C}_{B_\sigma}<0$. 现证明$\mathcal{C}_E>-\infty$.

情形1：$\frac{N+\alpha}{\alpha+2}\min\{\eta_1,\eta_2\}\geqslant-\lambda_1(\Omega)$.

对于$(\phi,\varphi)\in E$，可得

$$\begin{aligned}I(\phi,\varphi)&=I(\phi,\varphi)-\frac{1}{2\cdot 2_\alpha^*}I'(\phi,\varphi)(\phi,\varphi)\\&=\frac{\alpha+2}{2N+2\alpha}|\nabla\phi|_2^2-\frac{\alpha+2}{2N+2\alpha}\eta_1\int_\Omega(\phi^+)^2\log\left(\mathrm{e}^{\frac{\mu_1}{\eta_1}}(\phi^+)^2\right)\mathrm{d}x+\frac{\eta_1}{2}|\phi^+|_2^2\\&\quad+\frac{\alpha+2}{2N+2\alpha}|\nabla\varphi|^2-\frac{\alpha+2}{2N+2\alpha}\eta_2\int_\Omega(\varphi^+)^2\log\left(\mathrm{e}^{\frac{\mu_2}{\eta_2}}(\varphi^+)^2\right)\mathrm{d}x+\frac{\eta_2}{2}|\varphi^+|_2^2.\end{aligned}$$

既然$\eta_1<0$，再次使用$t\log t\geqslant-\mathrm{e}^{-1}$，推出

$$\begin{aligned}\frac{\alpha+2}{2N+2\alpha}\eta_1\int_\Omega(\phi_n^+)^2\log\left(\mathrm{e}^{\frac{\mu_1}{\eta_1}}(\phi_n^+)^2\right)\mathrm{d}x&\leqslant\frac{\alpha+2}{2N+2\alpha}\eta_1\int_{\mathrm{e}^{\frac{\mu_1}{\eta_1}}(\phi_n^+)^2\leqslant 1}(\phi_n^+)^2\log\left(\mathrm{e}^{\frac{\mu_1}{\eta_1}}(\phi_n^+)^2\right)\mathrm{d}x\\&\leqslant-\frac{\alpha+2}{2N+2\alpha}\eta_1\mathrm{e}^{-\frac{\mu_1}{\eta_1}}\int_{\mathrm{e}^{\frac{\mu_1}{\eta_1}}(\phi_n^+)^2\leqslant 1}\mathrm{e}^{-1}\mathrm{d}x\leqslant-\frac{\alpha+2}{2N+2\alpha}\eta_1\mathrm{e}^{-\frac{\mu_1}{\eta_1}-1}|\Omega|.\end{aligned}$$

类似地，由于$\eta_2<0$，我们有

$$\frac{\alpha+2}{2N+2\alpha}\eta_2\int_\Omega(\varphi_n^+)^2\log\left(\mathrm{e}^{\frac{\mu_2}{\eta_2}}(\varphi_n^+)^2\right)\mathrm{d}x\leq-\frac{\alpha+2}{2N+2\alpha}\eta_2\mathrm{e}^{-\frac{\mu_2}{\eta_2}-1}|\Omega|.$$

进一步

$$\frac{\alpha+2}{2N+2\alpha}|\nabla\phi|_2^2+\frac{\eta_1}{2}|\phi^+|_2^2\geq\left(\frac{\alpha+2}{2N+2\alpha}\lambda_1(\Omega)+\frac{\eta_1}{2}\right)|\phi^+|_2^2\geqslant 0.$$

同理可得

$$\frac{\alpha+2}{2N+2\alpha}|\nabla\varphi|_2^2+\frac{\eta_1}{2}|\varphi^+|_2^2\geqslant 0.$$

因此，推出

$$I(\phi,\varphi)\geqslant\frac{\alpha+2}{2N+2\alpha}\eta_1\mathrm{e}^{-\frac{\mu_1}{\eta_1}-1}|\Omega|+\frac{\alpha+2}{2N+2\alpha}\eta_2\mathrm{e}^{-\frac{\mu_2}{\eta_2}-1}|\Omega|>-\infty,$$

进而导出$\mathcal{C}_E>-\infty$.

情形2：当$\gamma > 0$时.

对任意$(\phi,\varphi)\in E$，我们得到

$$
\begin{aligned}
& I(\phi,\varphi)-\frac{1}{2}I'(\phi,\varphi)(\phi,\varphi)\\
&=\frac{\alpha+2}{2N+2\alpha}\int_\Omega\left(\beta_1\left(I_\alpha*|\phi^+|^{2_\alpha^*}\right)|\phi^+|^{2_\alpha^*}+\beta_2\left(I_\alpha*|\varphi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*}+2\gamma\left(I_\alpha*|\phi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*}\right)\mathrm{d}x+\frac{\eta_1}{2}|\phi^+|_2^2+\frac{\eta_2}{2}|\varphi^+|_2^2\\
&\geqslant\frac{\alpha+2}{2N+2\alpha}\int_\Omega\left(\beta_1\left(I_\alpha*|\phi^+|^{2_\alpha^*}\right)|\phi^+|^{2_\alpha^*}+\beta_2\left(I_\alpha*|\varphi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*}\right)\mathrm{d}x+\frac{\eta_1}{2}|\Omega|^{\frac{2}{2^*}}|\phi^+|_{2^*}^{2^*-2}+\frac{\eta_2}{2}|\Omega|^{\frac{2}{2^*}}|\varphi^+|_{2^*}^{2^*-2}.
\end{aligned}
$$

进而导出$\mathcal{C}_E>-\infty$.

情形3：$\gamma\in(-\sqrt{\beta_1\beta_2},0)$.

在此种情形下，对于耦合项，有如下估计：

$$
2\gamma\int_\Omega\left(I_\alpha*|\phi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*}\mathrm{d}x\geqslant\gamma\sqrt{\frac{\beta_1}{\beta_2}}\int_\Omega\left(I_\alpha*|\phi^+|^{2_\alpha^*}\right)|\phi^+|^{2_\alpha^*}\mathrm{d}x+\gamma\sqrt{\frac{\beta_2}{\beta_1}}\int_\Omega\left(I_\alpha*|\varphi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*}\mathrm{d}x.
$$

则对任意$(\phi,\varphi)\in E$，有

$$
\begin{aligned}
& I(\phi,\varphi)-\frac{1}{2}I'(\phi,\varphi)(\phi,\varphi)\\
&\geqslant\frac{\alpha+2}{2N+2\alpha}\left(\left(\beta_1+\gamma\sqrt{\frac{\beta_1}{\beta_2}}\right)\int_\Omega\left(I_\alpha*|\phi^+|^{2_\alpha^*}\right)|\phi^+|^{2_\alpha^*}\mathrm{d}x+\left(\beta_2+\gamma\sqrt{\frac{\beta_2}{\beta_1}}\right)\int_\Omega\left(I_\alpha*|\varphi^+|^{2_\alpha^*}\right)|\varphi^+|^{2_\alpha^*}\mathrm{d}x\right)\\
&\quad+\frac{\eta_1}{2}|\Omega|^{\frac{2}{2^*}}|\phi^+|_{2^*}^{2^*-2}+\frac{\eta_2}{2}|\Omega|^{\frac{2}{2^*}}|\varphi^+|_{2^*}^{2^*-2}.
\end{aligned}
$$

从$\beta_1+\gamma\sqrt{\frac{\beta_1}{\beta_2}}>0$及$\beta_2+\gamma\sqrt{\frac{\beta_2}{\beta_1}}>0$中，导出$\mathcal{C}_E>-\infty$.

定理2的证明 对于$\mathcal{C}_E$，我们取极小化序列$\{(\phi_n,\varphi_n)\}\subset E$，则$I'(\phi_n,\varphi_n)=0$. 根据引理3，可得$\{(\phi_n,\varphi_n)\}$在$\mathcal{H}$中有界. 故可假设

$$
(\phi_n,\varphi_n)\rightharpoonup(\phi,\varphi)\quad 在\ \mathcal{H}中.
$$

通过选取子列，我们也假设

$$
\begin{cases}
\phi_n\rightharpoonup\phi,\quad \varphi_n\rightharpoonup\varphi\quad 在\ L^{2^*}(\Omega)中,\\
\phi\to\phi,\quad \varphi_n\to\varphi\quad 在L^p(\Omega)中,\ 2\leqslant p<2^*,\\
\phi_n(x)\xrightarrow{a.e.}\phi(x),\quad \varphi_n(x)\xrightarrow{a.e.}\varphi(x)\quad 在\ \Omega\ 中.
\end{cases}
$$

与定理1的证明类似，可得$I'(\phi,\varphi)=0$，且$\phi\geqslant0$，$\varphi\geqslant0$. 令$u_n=\phi_n-\phi$及$v_n=\varphi_n-\varphi$，进而推出

$$
\begin{gathered}
|\nabla u_n|_2^2=\beta_1\int_\Omega\left(I_\alpha*|u_n^+|^{2_\alpha^*}\right)|u_n^+|^{2_\alpha^*}\mathrm{d}x+o_n(1),\qquad |\nabla v_n|_2^2=\beta_2\int_\Omega\left(I_\alpha*|v_n^+|^{2_\alpha^*}\right)|v_n^+|^{2_\alpha^*}\mathrm{d}x+o_n(1),\\
I(\phi_n,\varphi_n)=I(\phi,\varphi)+\frac{\alpha+2}{2N+2\alpha}\int_\Omega|\nabla u_n|^2\mathrm{d}x+\frac{\alpha+2}{2N+2\alpha}\int_\Omega|\nabla v_n|^2\mathrm{d}x+o_n(1).
\end{gathered}\tag{10}
$$

在子列的意义下，假设

$$
\int_\Omega|\nabla u_n|^2\mathrm{d}x=l_1+o_n(1),\qquad \int_\Omega|\nabla v_n|^2\mathrm{d}x=l_2+o_n(1).
$$

在式(10)中令$n\to\infty$,有

$$
\mathcal{C}_E\leqslant I(\phi,\varphi)\leqslant I(\phi,\varphi)+\frac{\alpha+2}{2N+2\alpha}l_1+\frac{\alpha+2}{2N+2\alpha}l_2=\lim_{n\to\infty}I(\phi_n,\varphi_n)=\mathcal{C}_E,
$$

从而导出$l_1=l_2=0$.因此，至多选取子列，可推出

$$
(\phi_n,\varphi_n)\to(\phi,\varphi)\quad 在\ \mathcal{H}中.
$$

由于$I(\phi,\varphi)=\mathcal{C}_E<0$，则有$(\phi,\varphi)\neq(0,0)$. 得证.

3 结 语

本文研究一类带有对数项和临界指数的耦合椭圆系统，是对经典的Sobolev临界问题在Hardy-Littlewood-Sobolev意义下的推广和改进.由于卷积项的出现，对其收敛性和能量估计，需要较多的分析技巧，本文的结果扩展了近年来有关耦合Choquard型系统的存在性定理(Sang,2021).

参考文献：

ALVES C O，FIGUEIREDO G M，MOLLE R，2021. Multiple positive bound state solutions for a critical Choquard equation[J]. Discrete Contin Dyn Syst，41(10)：4887–4919.

CINGOLANI S，GALLO M，TANAKA K，2022. On fractional Schrödinger equations with Hartree type nonlinearities[J]. Math Eng，4(6)：2022056.

DENG Y B，HE Q H，PAN Y Q，et al，2023. The existence of positive solution for an elliptic problem with critical growth and logarithmic perturbation[J]. Adv Nonlinear Stud，23(1)：20220049.

GAO F S，YANG M B，2018. The Brezis-Nirenberg type critical problem for the nonlinear Choquard equation[J]. Sci China Math，61(7)：1219–1242.

GENG Q P，TU Y Y，WANG J，2022. Existence and multiplicity of the positive normalized solutions to the coupled Hartree-Fock type nonlocal elliptic system [J]. J Fixed Point Theory Appl，24：83.

GHIMENTI M，Van SCHAFTINGEN J，2016. Nodal solutions for the Choquard equation[J]. J Funct Anal，271(1)：107–135.

GIACOMONI J，MUKHERJEE T，SREENADH K，2018. Doubly nonlocal system with Hardy-Littlewood-Sobolev critical nonlinearity [J]. J Math Anal Appl，467(1)：638–672.

HAJAIEJ H，LIU T H，SONG L J，et al，2024a. Positive solution for an elliptic system with critical exponent and logarithmic terms [J]. J Geom Anal，34(6)：182.

HAJAIEJ H，LIU T H，ZOU W M，2024b. Positive solution for an elliptic system with critical exponent and logarithmic terms：The higher-dimensional cases [J]. J Fixed Point Theory Appl，26：11.

HE Q H，HE Y，LV J T，2023. The existence of positive solutions to the Choquard equation with critical exponent and logarithmic term [J]. J Math Anal Appl，519(1)：126737.

LI Q，HAN Y Z，WANG T L，2023. Existence and nonexistence of solutions to a critical biharmonic equation with logarithmic perturbation [J]. J Differ Equ，365：1–37.

LI X F，2023. Nonexistence，existence and symmetry of normalized ground states to Choquard equations with a local perturbation [J]. Complex Var Elliptic Equ，68(4)：578–602.

SANG Y B，2021. Critical Kirchhoff-Choquard system involving the fractional p-Laplacian operator and singular nonlinearities[J]. Topol Methods Nonlinear Anal，58(1)：233–274.

XIA J K，ZHANG X，2021. Saddle solutions for the critical Choquard equation[J]. Calc Var Partial Differ Equ，60：53.

YOU S，ZHAO P H，WANG Q X，2019. Positive ground states for coupled nonlinear Choquard equations involving Hardy-Littlewood-Sobolev Critical Exponent [J]. Nonlinear Anal，Real World Appl，48：182–211.

ZHANG H，ZHANG J J，ZHONG X X，2023. Normalized solutions for critical Choquard sytems[EB/OL]. arXiv：2307.01483. http://arxiv.org/abs/2307.01483

ZHANG Q，HAN Y Z，WANG J，2023. A note on a critical bi-harmonic equation with logarithmic perturbation[J]. Appl Math Lett，145：108784.

ZHENG Y，SANTOS C A，SHEN Z F，et al，2020. Least energy solutions for coupled Hartree system with Hardy-Littlewood-Sobolev critical exponents [J]. Commun Pure Appl Anal，19(1)：329–369.

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